Approximate a function within the given interval with orthogonal polynomials.
Instructor: Nam Sun Wang

Given Function defined over x=[x_{lo},x_{hi}] $x_{lo} := 0$ $x_{hi} := 5$

$$f(x) := x \cdot e^{-x}$$

Taylor's Series Expansion around $x := 0$ for $i := 0 .. 4$ terms.

$$a_i := \frac{d^i}{d x^i} f(x) \qquad a^T = \begin{pmatrix} 0 & 1 & -2 & 3 & -4 \end{pmatrix} \qquad f_{Taylor}(x) := \sum_{i=0}^{4} \frac{a_i}{i!} \cdot x^i$$

Let us approximate the same function with orthogonal polynomials. We need to define the inner product between two functions (and the corresponding magnitude and angle).

$$\text{prod}(f,g) := \int_{-1}^{1} f(z) \cdot g(z)\, dz \qquad \text{mag}(f) := \sqrt{\text{prod}(f,f)} \qquad \theta(f,g) := \text{acos}\left(\frac{\text{prod}(f,g)}{\text{mag}(f) \cdot \text{mag}(g)}\right)$$

Consider Legendre polynomials, which is orthogonal over z=[-1,1]

$$P_0(z) := 1 \quad P_1(z) := z \quad P_2(z) := \frac{1}{2} \cdot \left(3 \cdot z^2 - 1\right) \quad P_3(z) := \frac{1}{2} \cdot \left(5 \cdot z^3 - 3 \cdot z\right) \quad P_4(z) := \frac{1}{8} \cdot \left(35 \cdot z^4 - 30 \cdot z^2 + 3\right)$$

Check: $\theta\left(P_0, P_1\right) = 90 \cdot \text{deg}$ $\theta\left(P_0, P_2\right) = 90 \cdot \text{deg}$ $\theta\left(P_0, P_3\right) = 90 \cdot \text{deg}$ $\theta\left(P_0, P_4\right) = 90 \cdot \text{deg}$

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$\theta\left(P_2, P_3\right) = 90 \cdot \text{deg}$ $\theta\left(P_2, P_4\right) = 90 \cdot \text{deg}$

$\theta\left(P_3, P_4\right) = 90 \cdot \text{deg}$

Now, let us consider a new interval of x=[0,5].

$$\text{prod}(f,g) := \int_{x_{lo}}^{x_{hi}} f(x) \cdot g(x)\, dx \qquad \text{mag}(f) := \sqrt{\text{prod}(f,f)} \qquad \theta(f,g) := \text{acos}\left(\frac{\text{prod}(f,g)}{\text{mag}(f) \cdot \text{mag}(g)}\right)$$

We obtain the orthogonal Legendre polynomial over this new interval by a change of variable, i.e., by replacing z with 0.4*x-1 in the Legendre polynomial originally defined over x=[-1,1].

$$z = \frac{x - \dfrac{x_{hi} + x_{lo}}{2}}{\left(\dfrac{x_{hi} - x_{lo}}{2}\right)} = 0.4 \cdot x - 1$$

$$P_0(x) := 1 \qquad P_1(x) := 0.4 \cdot x - 1 \qquad P_2(x) := \frac{1}{2} \cdot \left[3 \cdot (0.4 \cdot x - 1)^2 - 1\right]$$

$$P_3(x) := \frac{1}{2} \cdot \left[5 \cdot (0.4 \cdot x - 1)^3 - 3 \cdot (0.4 \cdot x - 1)\right] \qquad P_4(x) := \frac{1}{8} \cdot \left[35 \cdot (0.4 \cdot x - 1)^4 - 30 \cdot (0.4 \cdot x - 1)^2 + 3\right]$$

The same set of functions in vector notation: $P(x) := \left(P_0(x) \;\; P_1(x) \;\; P_2(x) \;\; P_3(x) \;\; P_4(x) \right)^T$

Check: $\theta(P_0, P_1) = 90 \cdot deg$ $\quad \theta(P_0, P_2) = 90 \cdot deg$ $\quad \theta(P_0, P_3) = 90 \cdot deg$ $\quad \theta(P_0, P_4) = 90 \cdot deg$

$\theta(P_1, P_2) = 90 \cdot deg$ $\quad \theta(P_1, P_3) = 90 \cdot deg$ $\quad \theta(P_1, P_4) = 90 \cdot deg$

$\theta(P_2, P_3) = 90 \cdot deg$ $\quad \theta(P_2, P_4) = 90 \cdot deg$

$\theta(P_3, P_4) = 90 \cdot deg$

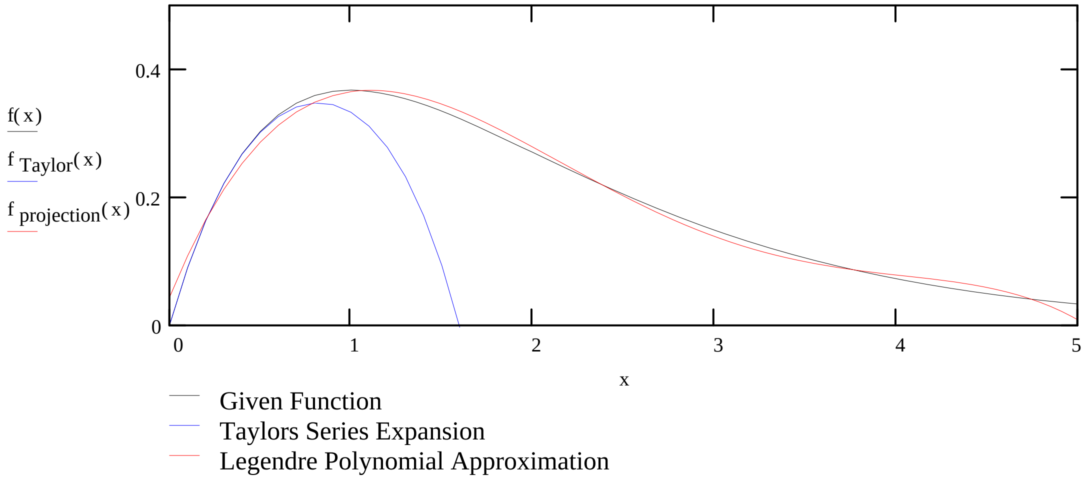
The best approximation of f in the space spanned by the Legendre polynomials is the **projection** of the function f into the Legendre polynomial space.

$$b_0 := \frac{prod(f, P_0)}{prod(P_0, P_0)} \quad b_1 := \frac{prod(f, P_1)}{prod(P_1, P_1)} \quad b_2 := \frac{prod(f, P_2)}{prod(P_2, P_2)} \quad b_3 := \frac{prod(f, P_3)}{prod(P_3, P_3)} \quad b_4 := \frac{prod(f, P_4)}{prod(P_4, P_4)}$$

$$f_{projection}(x) := \sum_{i=0}^{4} b_i \cdot P(x)_i$$

Comparison of Taylor's series expansion and the orthogonal polynomial approximation over $x=[x_{lo}, x_{hi}]$

$$x := x_{lo}, x_{lo} + 0.1 .. x_{hi}$$



We can see from the above plot that the Taylor's series expansion is valid only in the close vicinity of x=0 and quickly deviates from the original function it is trying to approximate. On the other hand, approximation with orthogonal Legendre polynomials of the same degree as the Taylor's series is quite good over the entire range.

Now, let us do a least squares approximation of f(x) with five basis functions 1, x, x^2, x^3, and x^4.

There are five basis functions: $i := 0 .. 4$

Let's evaluate f at 51 points: $j := 0 .. 50$ $\quad xx_j := 0.1 \cdot j$

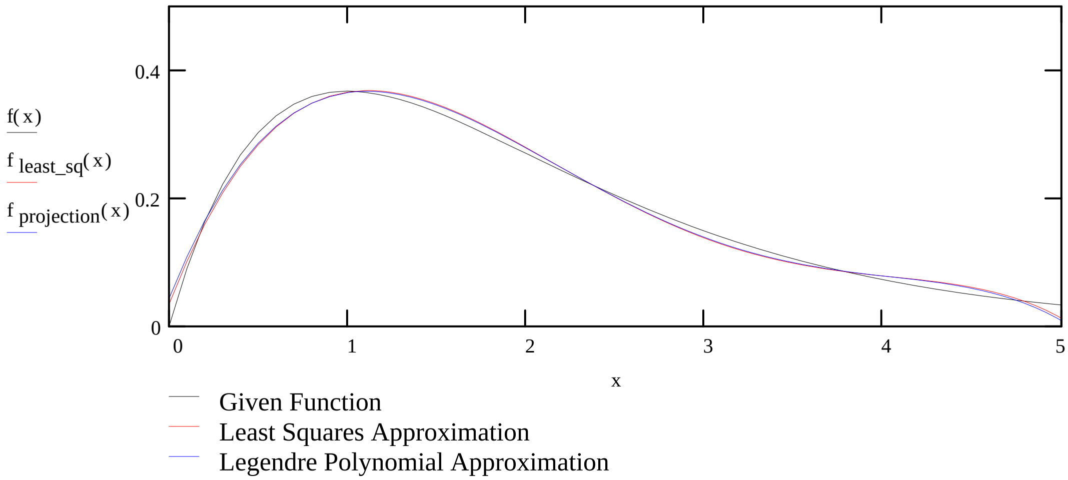
$$X_{j,i} := (xx_j)^i$$

$$y_j := f(xx_j)$$

Least squares formula: $c := (X^T \cdot X)^{-1} \cdot X^T \cdot y$

$$f_{least\_sq}(x) := \sum_{i=0}^{4} c_i \cdot x^i$$

Comparison of the least squares approximation and the orthogonal polynomial approximation over $x=[x_{lo},x_{hi}]$



It appears that the projection method with Legendre polynomials and the least squares method give identical results.