Inner product and angle between two vectors. Note that a function is a special case of a vector.

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Define **angle** between two vectors x & y: $x \cdot y = |x| \cdot |y| \cdot \cos(\theta)$

$$\theta(x,y) := \operatorname{acos}\left(\frac{x \cdot y}{|x| \cdot |y|}\right)$$

Projection of x into y is the inner product of (x,y) in the direction of y.

$$\text{projection}(x,y) := \frac{(x \cdot y) \cdot y}{(|y|)^2}$$

Example:

$$x := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad y := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Mathcad note: there is a "." after x, so that we can do symbolic evaluation later on this worksheet.

$$\theta(x.,y) = 54.736 \cdot \text{deg}$$

Schwarz inequality: $(|x.\cdot y| \le |x.| \cdot |y|) = 1$ "1" for "true"

Triangular inequality: $(|x. + y| \le |x.| + |y|) = 1$

$$\text{projection}(x.,y) = \begin{pmatrix} 0.333 \\ 0.333 \\ 0.333 \end{pmatrix} \qquad \sum \text{projection}(x.,y) = 1 \qquad |\text{projection}(x.,y)| = 0.577$$

Cross product (torque = radius x force)

$$\text{torque}(r,F) := r \times F$$

$$\phi(r,F) := \operatorname{asin}\left(\frac{|r \times F|}{|r| \cdot |F|}\right)$$

An example:

$$r := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad F := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \text{torque}(r,F) = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \qquad \phi(r,F) = 54.736 \cdot \text{deg}$$

Define **inner product** between two functions (and the definition of magnitude naturally follows)

$$\mathrm{prod}(f,g) := \int_{-1}^{1} f(x)\cdot g(x)\,dx \qquad \mathrm{mag}(f) := \sqrt{\mathrm{prod}(f,f)} \qquad \theta(f,g) := \mathrm{acos}\left(\frac{\mathrm{prod}(f,g)}{\mathrm{mag}(f)\cdot \mathrm{mag}(g)}\right)$$

An example:

$$f_1(x) := 1 \qquad f_2(x) := x \qquad f_3(x) := x^2 \qquad f_4(x) := x^3$$

$\theta(f_1,f_2) = 90\cdot\mathrm{deg}$ $\quad\theta(f_1,f_3) = 41.81\cdot\mathrm{deg}$ $\quad\theta(f_1,f_4) = 90\cdot\mathrm{deg}$ $\quad$ 1 & x^2 are not orthogonal.

$\theta(f_2,f_3) = 90\cdot\mathrm{deg}$ $\quad\theta(f_2,f_4) = 23.578\cdot\mathrm{deg}$ $\quad$ x & x^3 are not orthogonal.

$\theta(f_3,f_4) = 90\cdot\mathrm{deg}$

Gram-Schmidt Process.

Problem Statement: Given power series f_j, construct orthogonal vectors/functions e_j. Below ee_j are the analytical results.

$e_1(x) := f_1(x) \qquad e_1(x) := 1 \qquad ee_1(x) := 1$

$$e_2(x) := f_2(x) - \frac{\mathrm{prod}(e_1,f_2)}{\mathrm{prod}(e_1,e_1)}\cdot e_1(x) \qquad \frac{\mathrm{prod}(e_1,f_2)}{\mathrm{prod}(e_1,e_1)} = 0$$

... o.k.

$e_2(x) \rightarrow x \qquad ee_2(x) := x$

Mathcad got overwhelmed between e_2 and e_3 and subsequently failed to calculate e_3 and e_4 correctly.

$\mathrm{prod}(e_1,f_3) = 0.667$ $\quad \dfrac{\mathrm{prod}(e_1,f_3)}{\mathrm{prod}(e_1,e_1)} = 0.333$ $\quad \mathrm{prod}(e_2,f_3) = 2.974\cdot 10^{-4}$ $\quad \dfrac{\mathrm{prod}(e_2,f_3)}{\mathrm{prod}(e_2,e_2)} = 1$

$\mathrm{prod}(e_1,e_1) = 2$ $\quad \mathrm{prod}(e_2,e_2) = 2.974\cdot 10^{-4}$

The ratio of the inner product in the second term should have been 0, not 1:

$\mathrm{prod}(ee_2,f_3) = 0$ $\quad \dfrac{\mathrm{prod}(ee_2,f_3)}{\mathrm{prod}(ee_2,ee_2)} = 0$

$\mathrm{prod}(ee_2,ee_2) = 0.667$

$$e_3(x) := f_3(x) - \frac{\mathrm{prod}(e_1,f_3)}{\mathrm{prod}(e_1,e_1)}\cdot e_1(x) - \frac{\mathrm{prod}(e_2,f_3)}{\mathrm{prod}(e_2,e_2)}\cdot e_2(x)$$

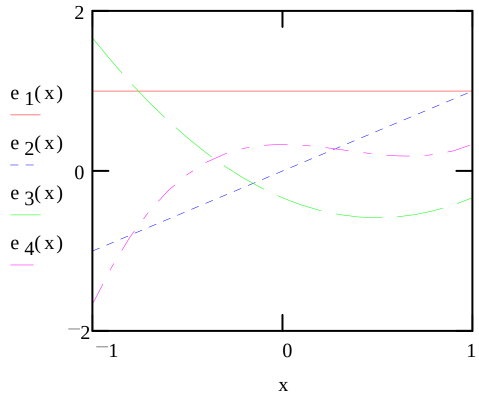
$e_3(x) \rightarrow x^2 - \dfrac{1}{3} \qquad ee_3(x) := x^2 - \dfrac{1}{3}$

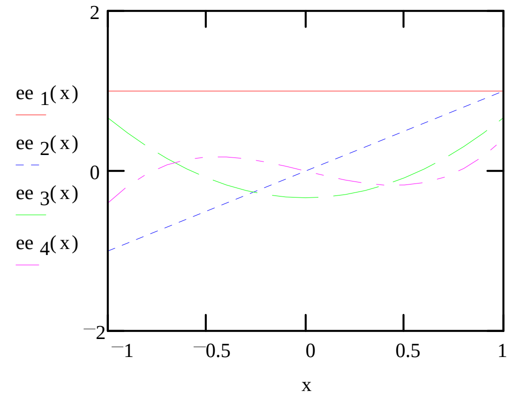
$$e_4(x) := f_4(x) - \frac{\mathrm{prod}(e_1,f_4)}{\mathrm{prod}(e_1,e_1)}\cdot e_1(x) - \frac{\mathrm{prod}(e_2,f_4)}{\mathrm{prod}(e_2,e_2)}\cdot e_2(x) - \frac{\mathrm{prod}(e_3,f_4)}{\mathrm{prod}(e_3,e_3)}\cdot e_3(x)$$

$e_4(x) \rightarrow x^3 - \dfrac{3}{5}\cdot x \qquad ee_4(x) := x^3 - \dfrac{3}{5}\cdot x$

$x := -1, -0.9 .. 1$

Comparison of numerical results and analytical results.





Check: (Numerical results are no good.)

$\theta(e_1, e_2) = 90 \cdot deg$ $\theta(e_1, e_3) = 89.177 \cdot deg$ $\theta(e_1, e_4) = 90 - 0.878i \cdot deg$

$\theta(e_2, e_3) = 27.937 \cdot deg$ $\theta(e_2, e_4) = 90 + 48.149i \cdot deg$

$\theta(e_3, e_4) = 90 + 53.176i \cdot deg$

Check: (Analytical results are o.k.)

$\theta(ee_1, ee_2) = 90 \cdot deg$ $\theta(ee_1, ee_3) = 90 \cdot deg$ $\theta(ee_1, ee_4) = 90 \cdot deg$

$\theta(ee_2, ee_3) = 90 \cdot deg$ $\theta(ee_2, ee_4) = 90 \cdot deg$

$\theta(ee_3, ee_4) = 90 \cdot deg$

Orthogonal polynomials (**Legendre Polynomials** .. un-normalized):

$P_0(x) := 1$ $P_1(x) := x$ $P_2(x) := \frac{1}{2} \cdot (3 \cdot x^2 - 1)$ $P_3(x) := \frac{1}{2} \cdot (5 \cdot x^3 - 3 \cdot x)$ $P_4(x) := \frac{1}{8} \cdot (35 \cdot x^4 - 30 \cdot x^2 + 3)$

$\theta(P_0, P_1) = 90 \cdot deg$ $\theta(P_0, P_2) = 90 \cdot deg$ $\theta(P_0, P_3) = 90 \cdot deg$ $\theta(P_0, P_4) = 90 \cdot deg$

$\theta(P_1, P_2) = 90 \cdot deg$ $\theta(P_1, P_3) = 90 \cdot deg$ $\theta(P_1, P_4) = 90 \cdot deg$

$\theta(P_2, P_3) = 90 \cdot deg$ $\theta(P_2, P_4) = 90 \cdot deg$

$\theta(P_3, P_4) = 90 \cdot deg$

Generating equation for Legendre polynomials:

$$P(x,n) := \text{if}\left(n = 1, 1, \frac{2 \cdot n - 1}{n} \cdot x \cdot P(x, n-1) - \frac{n-1}{n} \cdot P(x, n-2)\right)$$

The above is a recursive definition that does not work in version 5, but may work in version 6 (provided that we take care of n=0).

Legendre Polynomials of the Second Kind .. not quite mutually orthogonal based on the preceding definition of inner product:

$Q_0(x) := \ln\left(\sqrt{\frac{1+x}{1-x}}\right)$ $Q_1(x) := x \cdot Q_0(x) - 1$ $Q_2(x) := P_3(x) \cdot Q_0(x) - \frac{3}{2} \cdot x$

$\theta(Q_0, Q_1) = 90 \cdot deg$ $\theta(Q_0, Q_2) = 143.375 \cdot deg$ $\theta(Q_1, Q_2) = 71.86 \cdot deg$

For certain applications, a different interval is useful.

$$\mathrm{prod}(f,g) := \int_0^{2\cdot\pi} f(x)\cdot g(x)\,dx \qquad \mathrm{mag}(f) := \sqrt{\mathrm{prod}(f,f)}$$

$$\theta(f,g) := \mathrm{acos}\left(\frac{\mathrm{prod}(f,g)}{\mathrm{mag}(f)\cdot\mathrm{mag}(g)}\right)$$

An example:

$f(x) := \sin(x)$ $g(x) := \cos(x)$ $\theta(f,g) = 90\cdot\mathrm{deg}$ $\mathrm{mag}(f)^2 = 1\cdot\pi$ $\mathrm{mag}(g)^2 = 1\cdot\pi$

$f(x) := \sin(x)$ $g(x) := \sin(2\cdot x)$ $\theta(f,g) = 90\cdot\mathrm{deg}$

$f(x) := \sin(x)$ $g(x) := \sin(3\cdot x)$ $\theta(f,g) = 90\cdot\mathrm{deg}$

$f(x) := \cos(x)$ $g(x) := \cos(2\cdot x)$ $\theta(f,g) = 90\cdot\mathrm{deg}$

$f(x) := \sin(2\cdot x)$ $g(x) := \cos(3\cdot x)$ $\theta(f,g) = 90\cdot\mathrm{deg}$

$f(x) := \sin(x)$ $g(x) := \sin\left(x + \frac{\pi}{2}\right)$ $\theta(f,g) = 90\cdot\mathrm{deg}$ ⟵ Another interpretation of angle

$f(x) := \sin(x)$ $g(x) := \sin\left(x + \frac{\pi}{4}\right)$ $\theta(f,g) = 45\cdot\mathrm{deg}$ ⟵ Another interpretation of angle

Thus, sine and cosine functions have the nice property that they are orthogonal to each other.

For other applications, a weighting factor may be included.

$$w(x) := \frac{1}{\sqrt{1-x^2}}$$

$$\mathrm{prod}(f,g) := \int_{-1}^{1} w(x)\cdot f(x)\cdot g(x)\,dx \qquad \mathrm{mag}(f) := \sqrt{\mathrm{prod}(f,f)}$$

$$\theta(f,g) := \mathrm{acos}\left(\frac{\mathrm{prod}(f,g)}{\mathrm{mag}(f)\cdot\mathrm{mag}(g)}\right)$$

Example: Chebychev Polynomials.

$T_0(x) := 1$ $T_1(x) := x$ $T_2(x) := 2\cdot x^2 - 1$ $T_3(x) := 4\cdot x^3 - 3\cdot x$ $T_4(x) := 8\cdot x^4 - 8\cdot x^2 + 1$

$\theta(T_0, T_1) = 90\cdot\mathrm{deg}$ $\theta(T_0, T_2) = 90\cdot\mathrm{deg}$ $\theta(T_0, T_3) = 90\cdot\mathrm{deg}$ $\theta(T_0, T_4) = 90\cdot\mathrm{deg}$

$\theta(T_1, T_2) = 90\cdot\mathrm{deg}$ $\theta(T_1, T_3) = 90\cdot\mathrm{deg}$ $\theta(T_1, T_4) = 90\cdot\mathrm{deg}$

$\theta(T_2, T_3) = 90\cdot\mathrm{deg}$ $\theta(T_2, T_4) = 90\cdot\mathrm{deg}$

$\theta(T_3, T_4) = 90\cdot\mathrm{deg}$

Yet for other applications, a semi-infinite interval is useful, and a weighting factor may be included.

$$w(x) := \exp(-x)$$

$$\mathrm{prod}(f,g) := \int_0^{\infty} w(x)\cdot f(x)\cdot g(x)\,dx \qquad \mathrm{mag}(f) := \sqrt{\mathrm{prod}(f,f)}$$

$$\theta(f,g) := \mathrm{acos}\left(\frac{\mathrm{prod}(f,g)}{\mathrm{mag}(f)\cdot \mathrm{mag}(g)}\right)$$

Example: **Laguerre polynomials**.

$$L_0(x) := 1 \qquad L_1(x) := 1 - x \qquad L_2(x) := 2 - 4\cdot x + x^2$$

$$L_3(x) := 6 - 18\cdot x + 9\cdot x^2 - x^3 \qquad L_4(x) := 24 - 96\cdot x + 72\cdot x^2 - 16\cdot x^3 + x^4$$

Although the integrals do not converge numerically (because the upper intergration limit is ∞), they converge to 0 symbolically.

$$\int_0^{\infty} \exp(-x)\cdot L_0(x)\cdot L_1(x)\,dx =$$

not converging

$$\int_0^{\infty} \exp(-x)\cdot L_0(x)\cdot L_1(x)\,dx \rightarrow 0$$

$$\int_0^{\infty} \exp(-x)\cdot L_1(x)\cdot L_2(x)\,dx \rightarrow 0$$

The angles, calculated symbolically, are:

$\theta(L_0, L_1) \rightarrow \frac{1}{2}\cdot\pi$ $\quad \theta(L_0, L_2) \rightarrow \frac{1}{2}\cdot\pi$ $\quad \theta(L_0, L_3) \rightarrow \frac{1}{2}\cdot\pi$ $\quad \theta(L_0, L_4) \rightarrow \frac{1}{2}\cdot\pi$

$\theta(L_1, L_2) \rightarrow \frac{1}{2}\cdot\pi$ $\quad \theta(L_1, L_3) \rightarrow \frac{1}{2}\cdot\pi$ $\quad \theta(L_1, L_4) \rightarrow \frac{1}{2}\cdot\pi$

$\theta(L_2, L_3) \rightarrow \frac{1}{2}\cdot\pi$ $\quad \theta(L_2, L_4) \rightarrow \frac{1}{2}\cdot\pi$

$\theta(L_3, L_4) \rightarrow \frac{1}{2}\cdot\pi$

Yet for other applications, an infinite interval is useful, and a weighting factor may be included.

$$w(x) := \exp\left(-x^2\right)$$

$$\text{prod}(f,g) := \int_{-\infty}^{\infty} w(x)\cdot f(x)\cdot g(x)\,dx \qquad \text{mag}(f) := \sqrt{\text{prod}(f,f)}$$

$$\theta(f,g) := \text{acos}\left(\frac{\text{prod}(f,g)}{\text{mag}(f)\cdot\text{mag}(g)}\right)$$

Example: **Hermite polynomials**.

$$H_0(x) := 1 \quad H_1(x) := 2\cdot x \quad H_2(x) := 4\cdot x^2 - 2 \quad H_3(x) := 8\cdot x^3 - 12\cdot x \quad H_4(x) := 16\cdot x^4 - 48\cdot x^2 + 12$$

Although the integrals do not converge numerically (because the upper intergration limit is ∞), they converge to 0 symbolically.

$$\int_{-\infty}^{\infty} \exp\left(-z^2\right)\cdot H_0(z)\cdot H_1(z)\,dz =$$

overflow

$$\int_{-\infty}^{\infty} \exp\left(-z^2\right)\cdot H_0(z)\cdot H_1(z)\,dz \to 0$$

$$\int_{-\infty}^{\infty} \exp\left(-z^2\right)\cdot H_1(z)\cdot H_2(z)\,dz \to 0$$

The angles, calculated symbolically, are:

$$\theta\left(H_0, H_1\right) \to \frac{1}{2}\cdot\pi \quad \theta\left(H_0, H_2\right) \to \frac{1}{2}\cdot\pi \quad \theta\left(H_0, H_3\right) \to \frac{1}{2}\cdot\pi \quad \theta\left(H_0, H_4\right) \to \frac{1}{2}\cdot\pi$$

$$\theta\left(H_1, H_2\right) \to \frac{1}{2}\cdot\pi \quad \theta\left(H_1, H_3\right) \to \frac{1}{2}\cdot\pi \quad \theta\left(H_1, H_4\right) \to \frac{1}{2}\cdot\pi$$

$$\theta\left(H_2, H_3\right) \to \frac{1}{2}\cdot\pi \quad \theta\left(H_2, H_4\right) \to \frac{1}{2}\cdot\pi$$

$$\theta\left(H_3, H_4\right) \to \frac{1}{2}\cdot\pi$$